Exercise 7

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. If $\|\mathbf{v}\| = \|\mathbf{w}\|$, show that $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are orthogonal.

Solution

The aim is to show that the dot product of $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ is zero.

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) &= \mathbf{v} \cdot \mathbf{v} - \mathbf{v} \cdot \mathbf{w} + \mathbf{w} \cdot \mathbf{v} - \mathbf{w} \cdot \mathbf{w} \\ &= \|\mathbf{v}\|^2 - \cancel{\mathbf{w} \cdot \mathbf{v}} + \cancel{\mathbf{w} \cdot \mathbf{v}} - \|\mathbf{w}\|^2 \\ &= \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2\end{aligned}$$

If $\|\mathbf{v}\| = \|\mathbf{w}\|$, then

$$\begin{aligned}(\mathbf{v} + \mathbf{w}) \cdot (\mathbf{v} - \mathbf{w}) &= \|\mathbf{w}\|^2 - \|\mathbf{w}\|^2 \\ &= 0,\end{aligned}$$

which means $\mathbf{v} + \mathbf{w}$ and $\mathbf{v} - \mathbf{w}$ are orthogonal.